Kadec and Krein–Milman properties

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Abstract

The main goal of this paper is to prove that any Banach space X with the Krein–Milman property such that the weak and the norm topology coincide on its unit sphere admits an equivalent norm that is locally uniformly rotund.

Propriétés de Kadec et Krein–Milman

Résumé.

Le but de ce travail est de prouver que tout espace de Banach ayant la propriété de Krein–Milman et telle que la topologie faible et la topologie de la norme coincident sur la sphère unité, admet une norme équivalent localement uniformement convexe.

Version française abrégé

Une norme sur X est dite strictement convexe si S_X ne contient aucun segment non–trivial et il est dite localement uniformement convexe (**LUR** pour abréger) si $\|x_k - x\| \to 0$ lorsque $2\left(\|x_k\|^2 + \|x\|^2\right) - \|x_k + x\|^2 \to 0$.

Rappelons qu'un ensemble C d'un espace de Banach X a la propriété Krein–Milman si tout sous–ensemble de C fermé, convexe, borné et non vide a au moins un point extrémal. Nous dirons que la norme de X a la propriété de Krein–Milman si sa sphère unité S_X a la propriété de Krein–Milman ce qui est equivalent à dire que pour tout $f \in S_{X^*}$ l'ensemble convexe $\{x \in B_X : \ f(x) = 1\}$ a la propriété de Krein–Milman lorsqu'il n'est pas vide, c'est à dire si chaque face de S_X a la propriété de Krein–Milman.

Nous disons que la norme de X a la propriété de Kadec si l'application identité $(B_X, \text{faible}) \to (B_X, \|\cdot\|)$ est continue pour tous le points de S_X.

Dans [11] (voir aussi [2, p. 148] et [10, Théorème 1] pour une preuve élégante) est démontré que tout espace de Banach X ayant une norme qui soit strictement convexe et a la propriété de Kadec admet une norme **LUR** équivalent. Au [9] il fut démontré que X est renormable **LUR** si et seulement si pour tout $\varepsilon > 0$ est possible d'écrire $X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon}$ de telle manière que quel que soit $x \in X_{n,\varepsilon}$ il existe un demi-espace ouvert H contenant x avec

$$\text{diam } (X_{n,\varepsilon} \cap H) < \varepsilon.. \qquad (1)$$

L'idée de considérer une partition d'enombrable de X a son origine au [6] où de faibles voisinages sont apparus au lieu de demi–espaces ouverts. Étant donné que tout point extrémal de continuité est dentable (voir p.e. [7]) nous deduisons que le résultat [11] est une conséquence de [9].

C'est naturel se questionner si la propriété de Kadec implique l'existence d'une strictement convexe (ou maintenant cela revient à dire une **LUR**) norme équivalente. Nous trouvons la réponse négative donnée par R. Haydon [5]. Récemment au [10] le résultat de [9] a été étendu aux normes duales, c'est à dire, de normes équivalentes duales **LUR** ont été obtenues. Par conséquence nous obtenons que pour l'existence de telle norme il suffit que pour tout $\varepsilon > 0$ il existe une décomposition de X^* telle que le demi–espace H de (1) puisse être choisi comme un faible*–ouvert, (c'est à dire, il appartient à la famille $\mathcal{H}$ de tous les ensembles $\{f \in X^* : f(x) > \lambda\}$, $x \in X$, $\lambda \in \mathbb{R}$). Dans cet article nous obtenons une généralisation de ce résultat, en particulier nous prouvons.

Théorème *L'espace Y de Banach avec une norme laquelle a la propriété de Kadec et la propriété de Krein–Milman est renormable* **LUR**.

La norme est construit avec les metodes du M. Raja [10]. Certainement, pour $\varepsilon > 0$ nous choisons les ensembles $B_1(\varepsilon) = B_{Y^{**}}$, et $B_{n+1}(\varepsilon)$ formé avec tous les pointes $u \in B_n(\varepsilon)$ telle que $u \notin H$ si H est quelconque demi–espace faible*–ouvert dans Y^{**} avec $H \cap (2^{-1}B_{Y^{**}}) = \emptyset$ et $diam\,(B_n(\varepsilon) \cap B_Y \cap H) < \varepsilon$. Si $\|\cdot\|_{m,n}$ est le jauge de Minkowski pour l'ensemble $B_n\,(m^{-1})$ et

$$|y| = (\sum 2^{-m-n} \|y\|^2_{m,n})^{\frac{1}{2}}, \quad y \in Y,$$

nous avons le norme **LUR** dans Y.

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A norm in X is said to be rotund if S_X contains no non–trivial segment, and it is said to be locally uniformly rotund (**LUR** for short) if $\|x_k - x\| \to 0$ whenever $2\left(\|x_k\|^2 + \|x\|^2\right) - \|x_k + x\|^2 \to 0$. Let us recall that a set C in a Banach space X has the Krein–Milman property if every non empty convex bounded and norm–closed subset of C has at least one extreme point. We say that the norm of X has the Krein–Milman property if its unit sphere S_X has the Krein–Milman property which is equivalent to say that for every $f \in S_{X^*}$ the convex set $\{x \in B_X : \; f(x) = 1\}$ is either empty or has the Krein–Milman property. Let us mention that the Krein–Milman property of the norm is much weaker than the Krein–Milman property of the space. For instance every rotund norm has the Krein–Milman property despite the space may not have the Krein–Milman property.

Let $x \in C \subset X$, we say that x is a point of continuity for C if the identity map $(C, \text{weak}) \to (C, \|\cdot\|)$ is continuous at x. We say that the norm of X has the Kadec property if all points of S_X are points of continuity. In this paper we shall consider mainly weak* points of continuity and the weak* Kadec property that are those appearing when the weak topology above is replaced by the weak* topology.

In [11] (see also [2, p. 148] and [10, Theorem 1] for an elegant proof) it was shown that every Banach space X with a norm which is rotund and has the Kadec property admits an equivalent **LUR** norm. In [9] it was shown that X is **LUR** renormable if and only if for every $\varepsilon > 0$ it is possible to write $X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon}$ in such a way that for every $x \in X_{n,\varepsilon}$ there exists an open half space H containing x with

$$\text{diam } (X_{n,\varepsilon} \cap H) < \varepsilon. \tag{1}$$

The idea of considering a countable splitting of X goes back to [6] where weak neighbourhoods appeared instead of open half spaces. Since every extreme point of continuity is denting (see e.g. [7]) we obtain that the result of [11] is a consequence of [9].

It was natural to wonder whether the Kadec property implies the existence of a rotund (or now equivalently **LUR**) renorming. This was answered in the negative by R. Haydon [5]. Recently in [10] the result of [9] was extended to dual norms, i.e.

dual **LUR** renormings were obtained. It turned out that for the existence of such norms it is enough that for every $\varepsilon > 0$ there exists a countable decomposition of X^* such that the half space H in (1) can be chosen to be weak*–open, (i.e. it belongs to the family $\mathcal{H}$ of all sets $\{f \in X^* : f(x) > \lambda\}$, $x \in X$, $\lambda \in \mathbb{R}$). Using this result M. Raja [10] proved that X^* has a dual **LUR** norm provided the norm in X^* has the weak* Kadec property. In this paper using Raja's construction of the norm, we obtain a generalization of his result.

Theorem *Let X be a Banach (not necessarily Asplund) space and Y a subspace of X^*. If the norm of Y has the Krein–Milman property and all points of S_Y are weak* points of continuity for $B_Y \subset X^*$, then X^* admits an equivalent dual norm whose restriction to Y is* **LUR**. *In particular, every Banach space with a norm which has the Kadec property and the Krein–Milman property is* **LUR** *renormable.*

Remark In [3] it was shown that every dual of a Asplund space is **LUR** renormable. However there exists Banach spaces with the Radon–Nikodym property and so with the Krein–Milman property which cannot be isomorphically embeded in the dual of any Asplund space, [1], [8].

Lemma 1 *Let $(D_n)_1^\infty$, be a decreasing nested sequence of norm–closed and convex subsets of X^*, $D_\infty := \bigcap_1^\infty D_n$, $D := \bigcap_1^\infty \overline{D}_n^{w^*}$. If x is a weak* point of continuity for D_1 and an extreme point of D_∞ then x is an extreme point of D.*

Proof. Since x is a weak* point of continuity for D_1 we get that x is a weak* point of continuity for $\overline{D}_1^{w^*}$ and since $x \in D_\infty \subset D_n$ we have that x a weak* point of continuity for $\overline{D}_n^{w^*}$, $n = 1, 2, \ldots$

Take y, $z \in D$, $x = (y + z)/2$. Since x is a weak* point of continuity for $\overline{D}_n^{w^*}$ we can find a weak*–open absolutely convex neighbourhood of zero, U_n, such that

$$\lim_n \operatorname{diam}\left((x + U_n) \cap \overline{D}_n^{w^*}\right) = 0.$$

We show that for every $u \in (y + U_n) \cap \overline{D}_n^{w^*}$ we have $(u + z)/2 \in (x + U_n) \cap \overline{D}_n^{w^*}$. Indeed $(u + z)/2 - x = (u - y)/2 \in (1/2)U_n \subset U_n$ and since $\overline{D}_n^{w^*}$ is convex and $z \in D \subset \overline{D}_n^{w^*}$ we deduce $(u+z)/2 \in \overline{D}_n^{w^*}$. Consequently for all u, $v \in (y + U_n) \cap \overline{D}_n^{w^*}$ we have

$$\|u - v\|/2 = \left\| \frac{u + z}{2} - \frac{v + z}{2} \right\| \leq \operatorname{diam}\left((x + U_n) \cap \overline{D}_n^{w^*}\right).$$

So

$$\lim_n \operatorname{diam}\left((y + U_n) \cap \overline{D}_n^{w^*}\right) = 0. \tag{2}$$

Pick $w_n \in ((y + U_n) \cap D_n)$. From (2) it follows $\|w_n - y\| \to 0$. Since D_n are norm–closed we get $y \in \bigcap_1^\infty D_n = D_\infty$. This implies $x = y$. ■

Lemma 2 *Let Y be a subspace of X^* such that all points of S_Y are weak* points of continuity for $B_Y \subset X^*$, and the norm of Y has the Krein–Milman property. Let $\varepsilon > 0$, $\mathcal{H}$ be the family of all weak*–open half–spaces H of X^* such that $H \cap (2^{-1} B_{X^*}) = \emptyset$, $B_1(\varepsilon) = B_{X^*}$, and*

$$B_{n+1}(\varepsilon) = B_n(\varepsilon) \setminus \{u \in B_n(\varepsilon) \cap H : \ H \in \mathcal{H}, \ diam \ (B_n(\varepsilon) \cap B_Y \cap H) < \varepsilon\},$$

$n = 1, 2, \ldots$. Then $(\bigcap_1^\infty B_n(\varepsilon)) \cap S_Y = \emptyset$.

Proof. Set $B_n = B_n(\varepsilon)$. Assume the contrary and let $y \in (\cap_1^\infty B_n) \cap S_Y$. There exists $f \in S_{Y^*}$ with $f(y) = 1$. Set $F = \{u \in (\cap_1^\infty B_n) \cap S_Y : f(u) = 1\}$. Since S_Y has the Krein–Milman property we get that F has at least one extreme point, say v. Evidently v is an extreme point for $D_\infty := (\bigcap_1^\infty B_n) \cap B_Y$. By assumption v is a weak* point of continuity of B_Y. Taking into account that $\{B_n\}$ is decreasing from the previous lemma it follows that v is an extreme point of

$$D := \bigcap_1^\infty \left(\overline{B_n \cap B_Y}^{w^*} \right)$$

Since v is an extreme point of the weak*–compact set D according to Choquet Lemma (see e.g. [4, p. 49]) we get that the family $\mathcal{G} = \{H \cap D : \ H \in \mathcal{H}, \ v \in H\}$ forms a base of neighbourhoods for v in (D, weak^*). Since v is a weak* point of continuity for D we get that $\mathcal{G}$ forms a base of neighbourhoods for v in $(D, \|\cdot\|)$. So we can find $H_k \in \mathcal{H}$, $k = 1, 2, \ldots$ such that

$$v \in H_k \quad \text{and} \quad \text{diam} \left(\overline{H}_k^{w*} \cap D \right) \to 0. \tag{3}$$

Moreover we can find a strictly increasing sequence $\{n_k\}_1^\infty$ of positive integers and $H_k \in \mathcal{H}$, $k = 1, 2, \ldots$, such that

$$\overline{H}_{k+1}^{w*} \cap \overline{B_{n_{k+1}} \cap B_Y}^{w*} \subset \overline{H}_k^{w*} \cap \overline{B_{n_k} \cap B_Y}^{w*}. \tag{4}$$

Indeed, we can choose $H_{k+1} \in \mathcal{H}$ in such a way that $\overline{H}_{k+1}^{w*} \cap D \subset H_k \cap D$ so

$$\bigcap_{n > n_k + 1} \left(\overline{H}_{k+1}^{w*} \cap \overline{B_n \cap B_Y}^{w^*} \right) \subset H_k \cap D \subset H_k \cap \overline{B_{n_k} \cap B_Y}^{w^*}.$$

A compactness argument gives us an $n_{k+1} > n_k$ such that (4) is fulfilled.

On the other hand since

$$v \in B_{n_k+1} \subset B_{n_k} \setminus \{u \in B_{n_k} \cap H : \ H \in \mathcal{H}, \text{diam} \ (B_{n_k} \cap B_Y \cap H) < \varepsilon\}$$

we have diam $(B_{n_k} \cap B_Y \cap H_k) \geq \varepsilon$ and then we can pick

$$v_k \in (B_{n_k} \cap B_Y \cap H_k) \ \text{such that} \ \|v_k - v\| > \varepsilon/2. \tag{5}$$

According to (4) the sequence $\left\{\overline{H}_k^{w*} \cap \overline{B_{n_k} \cap B_Y}^{w*}\right\}_1^\infty$ is decreasing so any weak*–cluster point to the sequence $\{v_k\}_1^\infty$ must be in $\bigcap_1^\infty \left(\overline{H}_k^{w*} \cap \overline{B_{n_k} \cap B_Y}^{w*}\right) = \bigcap_1^\infty \left(\overline{H}_k^{w*} \cap D\right)$. From this observation and (3) it follows that v is the only weak*–cluster point to the sequence. Since v is a weak* point of continuity for B_Y we have that $\|v_k - v\| \to 0$ but this contradicts (5). ■

Proof of the theorem. This proof is a simple modification of [10, Theorem 1]. Let $B_n(\varepsilon)$ be from Lemma 2 and $\|\cdot\|_{m,n}$ be the Minkowski functional of $B_n(m^{-1})$. Set for $x \in X^*$

$$|x| = (\sum 2^{-m-n}\|x\|_{m,n}^2)^{\frac{1}{2}}.$$

We show that $|\cdot|$ is **LUR** on Y.

Let $x_k \in X^*$, $y \in S_Y$ and $2(|x_k|^2 + |y|^2) - |x_k + y|^2 \to 0$. By standard convex arguments we have for all $m, n \in \mathbb{N}$

$$\|x_k\|_{m,n}, \|(x_k + y)/2\|_{m,n} \to_k \|y\|_{m,n}. \tag{6}$$

Fix $m \in \mathbb{N}$. We show that $\overline{\lim}_k \|x_k - y\| \le 2m^{-1}$. Let $B_n = B_n(m^{-1})$. From Lemma 2 we have $(\cap_1^\infty B_n) \cap S_Y = \emptyset$. Since $S_Y \subset B_{X^*} = B_1$ and $y \in S_Y$ we can find n such that $y \in B_n \setminus B_{n+1}$. Then $\|y\|_{m,n} \le 1 < \|y\|_{m,n+1}$. Set $y_k = (\|y\|_{m,n}/\|x_k\|_{m,n})x_k$. From (6) we get that $\|x_k - y_k\| \to_k 0$ and that there exists k_m such that for $k > k_m$

$$\|y_k\|_{m,n} = \|y\|_{m,n} \le 1 < \|y_k\|_{m,n+1},$$

$$\|(y_k + y)/2\|_{m,n} \le (\|y_k\|_{m,n} + \|y\|_{m,n})/2 \le 1 < \|(y_k + y)/2\|_{m,n+1}.$$

Hence for $k > k_m$ we have $y_k, y, (y_k + y)/2 \in B_n \setminus B_{n+1}$. Pick $k > k_m$. From the definition of B_n follows that there exists a weak*–open half space H containing $(y_k + y)/2$ and $\operatorname{diam}(B_n \cap H) < m^{-1}$. Then either $y_k \in B_n \cap H$ or $y \in B_n \cap H$. Hence

$$\|y_k - y\|/2 = \min\{\|y_k - (y_k + y)/2\|, \|y - (y_k + y)/2\|\} \le \operatorname{diam}(B_n \cap H) < m^{-1}.$$

■

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